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Solution of a one-dimensional boson–fermion model

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Abstract. A one-dimensional boson–fermion model is considered. It is shown that the model is exactly solvable and the general Bethe eigenstates are constructed. On the basis of the Bethe ansatz equations, the ground state and the thermodynamics are also given in some closed integral equations.

1. Introduction

The Bethe ansatz [1] has proved to be a powerful method for tackling integrable models. Great achievements have been reached in the last few decades both in the (1+1)-dimensional quantum field theory [2, 3] and in condensed matter physics [4, 5]. Of the integrable family, a very special class is the N-wave interaction model and its various quantized versions. The study on the 1D N-wave interaction model has a long history. Exact results were obtained by many authors for both the classical cases [6, 7] and the quantum cases [8–10]. However, the quantum models considered in [8–10] are fundamentally unphysical for the ill-defined spectrum of the bosons.

In this paper, we consider a physically meaningful model. The Hamiltonian of our model reads

$$H = \int \Bigg\{ -\mathrm{i}v_{\mathrm{F}} \sum_{r=\pm,\sigma=\uparrow,\downarrow} rC^{\dagger}_{r,\sigma}(x)\partial_x C_{r,\sigma}(x) + g\sum_{\sigma}[b^{\dagger}_{\sigma}(x)C_{-,\sigma}(x)C_{+,\sigma}(x) + \mathrm{HC}]$$
$$+ \frac{g}{\sqrt{2}}[b^{\dagger}_{0}(x)(C_{-,\uparrow}(x)C_{+,\downarrow}(x) + C_{-,\downarrow}(x)C_{+,\uparrow}(x)) + \mathrm{HC}]\Bigg\}\,\mathrm{d}x \qquad (1)$$

where v_{F} is the Fermi velocity of the chiral fermions, $C^{\dagger}_{r,\sigma}(C_{r,\sigma})$ is the creation (annihilation) operator of the fermions with the chiral index r and the spin component σ, $b^{\dagger}_{\alpha}$ (b_{α}) $(\alpha = \sigma, 0)$ is the creation (annihilation) operator of the spin-1 bosons with the spin component α, and g is the boson–fermion coupling constant. Obviously, the interaction between the bosons and the fermions occurs only in the t-channel. Notice that although the bosons have no bare kinetic energy, the spontaneous decay into fermions does allow them to behave itinerantly. The factor $1/\sqrt{2}$ in the last term is introduced to ensure the two-fermion scattering matrix takes an universal form for the three triplets and thus the integrability of the model.

The contents of the present paper are as follows. In section 2 we construct the exact eigenstates of the Hamiltonian (1) with the coordinate Bethe ansatz. The Bethe ansatz equations are obtained from the periodic conditions of the Bethe wave functions. The ground-state properties are discussed in section 3. In section 4, we give the integral equations of the thermodynamics. Some concluding remarks are given in section 5.

2. Bethe states

To construct the eigenstates of the Hamiltonian, it is convenient to give some simple conserved quantities. From equation (1) we can see that the particle numbers

$$N_r = \int \left[\sum_\sigma C^\dagger_{r,\sigma}(x) C_{r,\sigma}(x) + \sum_\alpha b^\dagger_\alpha(x) b_\alpha(x) \right] \mathrm{d}x \tag{2}$$

$$M_\sigma = \int \left[\sum_r C^\dagger_{r,\sigma}(x) C_{r,\sigma}(x) + 2 b^\dagger_\sigma(x) b_\sigma(x) \right] \mathrm{d}x \tag{3}$$

and the total momentum

$$P = -\mathrm{i} \int \left[\sum_{r,\sigma} C^\dagger_{r,\sigma}(x) \partial_x C_{r,\sigma}(x) + \sum_\alpha b^\dagger_\alpha(x) \partial_x b_\alpha(x) \right] \mathrm{d}x \tag{4}$$

are conserved. Therefore, we may establish the common eigenstates of H, N_r, M_σ, and P. For simplicity, we denote the Bethe states by $|N_+, N_-\rangle$, but one should keep in mind that M_σ and P are also conserved in these states.

To show the procedure clearly, we first consider the $N_+ = N_- = 1$ case. In this case, the two fermions may form spin triplets or a spin singlet. Notice that only the spin triplets have non-trivial hybridization with the bosons. A spin triplet can be written as

$$\begin{aligned} |1,1\rangle_t = &\int \mathrm{d}x\ \mathrm{d}y\ \Psi_0^{\{\sigma_1,\sigma_2\}}(x,y)[C^\dagger_{+,\sigma_1}(x) C^\dagger_{-,\sigma_2}(y) + C_{+,\sigma_2}(x) C^\dagger_{-,\sigma_1}(y)]|0\rangle \\ &+ \int \mathrm{d}z\ \Psi_1^{\{\sigma_1,\sigma_2\}}(z) b^\dagger_{(\sigma_1+\sigma_2)/2}(z)|0\rangle \end{aligned} \tag{5}$$

where $(\uparrow + \downarrow)/2 = 0$ is supposed. Acting the Hamiltonian (1) on $|1,1\rangle_t$ we deduce the following Schrödinger equations

$$\begin{aligned} -\mathrm{i}v_\mathrm{F}(\partial_x - \partial_y)\Psi_0^{\{\sigma_1,\sigma_2\}}(x,y) + \tfrac{1}{2} g \Psi_1^{\{\sigma_1,\sigma_2\}}(x)\delta(x-y)[\delta_{\sigma_1,\sigma_2} + \sqrt{2}\delta_{\sigma_1,-\sigma_2}] \\ = E\Psi_0^{\{\sigma_1,\sigma_2\}}(x,y) \end{aligned} \tag{6}$$

$$\{2\delta_{\sigma_1,\sigma_2} + \sqrt{2}\delta_{\sigma_1,-\sigma_2}\} g \Psi_0^{\{\sigma_1,\sigma_2\}}(x,x) = E\Psi_1^{\{\sigma_1,\sigma_2\}}(x)$$

where E is the eigenvalue. The above equations can be solved by the following ansatz:

$$\begin{aligned} \Psi_0^{\{\sigma_1,\sigma_2\}}(x,y) &= \mathrm{e}^{\mathrm{i}kx+\mathrm{i}qy}[S_t(k-q)\theta(y-x) + \theta(x-y)] \\ \Psi_1^{\{\sigma_1,\sigma_2\}}(x) &= S^+_{\sigma_1,\sigma_2}(k-q)\Psi_0^{\{\sigma_1,\sigma_2\}}(x,x) \end{aligned} \tag{7}$$

with

$$\begin{aligned} S_t(k-q) &= \frac{k-q+\mathrm{i}c}{k-q-\mathrm{i}c} \\ S^+_{\sigma_1,\sigma_2}(k-q) &= \frac{g/v_\mathrm{F}\{2\delta_{\sigma_1,\sigma_2} + \sqrt{2}\delta_{\sigma_1,-\sigma_2}\}}{k-q} \\ E &= v_\mathrm{F}(k-q) \end{aligned} \tag{8}$$

where $c = g^2/4v_F$. Now we clear that the factor $1/\sqrt{2}$ in the last term of the Hamiltonian is to ensure the spin $SU(2)$ invariant and the universal form of the scattering matrix $S_t(k-q)$ for the three triplets. Since the spin singlet state does not contain boson, we can easily deduce its scattering matrix to be $S_s = 1$. Thus the scattering matrix of two electrons with different moving directions takes the form

$$S_{r,-r}(k-q) = \frac{k-q+\mathrm{i}cp_{12}}{k-q-\mathrm{i}c} \tag{9}$$

where $p_{12} = 1$ for spin triplets and $p_{12} = -1$ for a spin singlet. In fact, p_{12} is the eigenvalue of the spin exchange operator P_{12}. We proceed to discuss the $N_r = 2$, $N_{-r} = 0$ case. The eigenstates for $N_+ = 2$, $N_- = 0$ can be written as

$$|2, 0\rangle = \int \mathrm{d}x_1\, \mathrm{d}x_2\; \Psi^{\{\sigma_1,\sigma_2\}}(x_1, x_2) C_{+,\sigma_1}(x_1) C_{+,\sigma_2}(x_2)|0\rangle\,. \tag{10}$$

The Schrödinger equations thus reads

$$-i(\partial_{x_1} + \partial_{x_2})\Psi^{\{\sigma_1,\sigma_2\}}(x_1, x_2) = E\Psi^{\{\sigma_1,\sigma_2\}}(x_1, x_2)\,. \tag{11}$$

The above equation has a general solution

$$\begin{aligned}\Psi^{\{\sigma_1,\sigma_2\}}(x_1, x_2) &= \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2}[S_{+,+}(k_1-k_2)\theta(x_2-x_1) + \theta(x_1-x_2)]\\ &\quad +\mathrm{e}^{\mathrm{i}k_1x_2+\mathrm{i}k_2x_1}[S_{+,+}(k_1-k_2)\theta(x_1-x_2) + \theta(x_2-x_1)] \end{aligned}\tag{12}$$

$$E = v_F(k_1 + k_2)\,.$$

Also, these states do not involve bosons. Since the Schrödinger equation is only a first-order differential equation, the scattering matrix can be chosen arbitrarily. However, to ensure integrability, the Yang–Baxter equation

$$S_{r_1,r_2}(k_1,k_2)S_{r_1,r_3}(k_1,k_3)S_{r_2,r_3}(k_2,k_3) = S_{r_2,r_3}(k_2,k_3)S_{r_1,r_3}(k_1,k_3)S_{r_1,r_2}(k_1,k_2) \tag{13}$$

must be satisfied. A natural choice of $S_{r,r}$ is thus

$$S_{r,r}(k_1-k_2) = \mathrm{e}^{\mathrm{i}\phi_r(k_1-k_2)}\frac{k_1-k_2+\mathrm{i}cp_{12}}{k_1-k_2+\mathrm{i}c}\,. \tag{14}$$

The phase factor $\phi_r(k)$ is real for k real. It cannot be determined uniquely for the linear dispersion relation of the present model.

We now turn to the construction of the general eigenstates with arbitrary N_r. For simplicity, we consider only the highest-weight solutions in the spin sector. Since the total spin is a good quantum number, we can use the spin-flip operator

$$F = \int \left\{\sum_r C^\dagger_{r,\downarrow}(x)C_{r,\uparrow}(x) + \sqrt{2}[b^\dagger_0(x)b_\uparrow(x) + b^\dagger_\downarrow(x)b_0(x)]\right\}\mathrm{d}x \tag{15}$$

to obtain the other eigenstates. Note $[F, H] = 0$. For the highest-weight states, the quantities

$$N_{r\sigma} = \int \{C^\dagger_{r,\sigma}(x)C_{r,\sigma}(x) + b^\dagger_\sigma(x)b_\sigma(x)\}\,\mathrm{d}x \tag{16}$$

are also conserved. Thus the general highest-weight eigenstate $|N_+, N_-\rangle$ can be written as

$$|N_+, N_-\rangle = \sum_{m_\uparrow=0}^{\min[N_{r,\uparrow}]}\sum_{m_\downarrow=0}^{\min[N_{r,\downarrow}]}\Psi^{\{\sigma\}_\pm}_{m_\uparrow,m_\downarrow}(x_1,\ldots,x_{N_+-M}|y_1,\ldots,y_{N_--M}|z_1,\ldots,z_{m_\uparrow}|w_1,\ldots,w_{m_\downarrow})$$

$$\times\phi^{\{\sigma\}_\pm}_{m_\uparrow,m_\downarrow}\prod_{i=1}^{N_+-M}C^\dagger_{+,\sigma_i}(x_i)\,\mathrm{d}x_i\prod_{j=1}^{N_--M}C^\dagger_{-,\sigma_{N_++j}}(y_j)\,\mathrm{d}y_j$$

$$\times\prod_{l=1}^{m_\uparrow}b^\dagger_\uparrow(z_l)\,\mathrm{d}z_l\prod_{m=1}^{m_\downarrow}b^\dagger_\downarrow(w_m)\,\mathrm{d}w_m|0\rangle \tag{17}$$

where $M=m_\uparrow+m_\downarrow$ and $|0\rangle$ is the pseudo-vacuum defined by $C_{r,\sigma}|0\rangle=b_\alpha|0\rangle=0$. Notice that the $C^\dagger_{r,\sigma}$ in the products have a well-defined order. That means that if $i<i'$, $C^\dagger_{r,\sigma_i}$ must be in the left side of $C^\dagger_{r,\sigma_{i'}}$. The symbol $\{\sigma\}_\pm$ indicates all possible choices of the spin configurations with the total spin conserved in each channel r. The prefactor $\phi^{\{\sigma\}_\pm}_{m_\uparrow,m_\downarrow}$ is introduced to cancel the repeat of the 'fusion–decay processes':

$$\phi^{\{\sigma\}_\pm}_{m_\uparrow,m_\downarrow}=\left\{\prod_\sigma\left[m_\sigma!\prod_r N_{r\sigma}!\right]\right\}^{-1}. \tag{18}$$

In what follows we shall omit the superscript $\{\sigma\}_\pm$, but one should keep in mind that the total spin of the wave function is conserved. The Schrödinger equation in the first quantization form is

$$\begin{aligned}
&-\mathrm{i}v_\mathrm{F}\Big(\sum_i\partial_{x_i}-\sum_j\partial_{y_j}\Big)\Psi_{m_\uparrow,m_\downarrow}(x_1,\ldots,x_{N_+-M}|y_1,\ldots,y_{N_--M}|z_1,\ldots,z_{m_\uparrow}|w_1,\ldots,w_{m_\downarrow})\\
&\qquad+g\sum_{ij}R_{ij}\Psi_{m_\uparrow+1,m_\downarrow}(\cdots,x_{i-1},x_{i+1},\ldots|\ldots,y_{j-1},y_{j+1},\ldots|x_i,\ldots|\cdots)\\
&\qquad\times\delta(x_i-y_j)\\
&\qquad+g\sum_{i'j'}R_{i'j'}\Psi_{m_\uparrow,m_\downarrow+1}(\ldots,x_{i'-1},x_{i'+1},\ldots|\ldots,y_{j'-1},y_{j'+1},\ldots|\cdots|x_{i'},\ldots)\\
&\qquad\times\delta(x_{i'}-y_{j'})\\
&\qquad+g\sum_{l=1}^{m_\uparrow}\Psi_{m_\uparrow-1,m_\downarrow}(\ldots,z_l|z_l,\ldots|\ldots,z_{l-1},z_{l+1},\ldots|\cdots)\\
&\qquad+g\sum_{m=1}^{m_\downarrow}\Psi_{m_\uparrow,m_\downarrow-1}(\cdots,w_m|w_m,\ldots|\cdots|\ldots,w_{m-1},w_{m+1},\ldots)\\
&\quad=E\Psi_{m_\uparrow,m_\downarrow}
\end{aligned} \tag{19}$$

where $R_{ij}=(-1)^{N_+-M+i+j-1}\delta_{\sigma_i,\sigma_{N_++j}}$.

From the experience of the two-body case, we know that the $\Psi_{m_\uparrow,m_\downarrow}$ are the 'contraction' of $\Psi_{0,0}$ [11] which takes the form

$$\Psi_{0,0}(x_1,\ldots,x_{N_+}|y_1,\ldots,y_{N_-})=\sum_{P,Q}A_P(Q)\exp\left[\mathrm{i}\sum_{j=1}^N s_{P_j}t_{Q_j}\right]\prod_{j=1}^N\delta^{\gamma_{P_j}}_{r_{Q_j}}\theta(t_{Q_1}<\cdots<t_{Q_N}) \tag{20}$$

where $N=N_++N_-$; P,Q are the permutations of $(1,\ldots,N)$ and

$$\{s_1,\ldots,s_N\}=\{k_1,\ldots,k_{N_+};q_1,\ldots,q_{N_-}\}\qquad\{t_1,\ldots,t_N\}=\{x_1,\ldots,x_{N_+};y_1,\ldots,y_{N_-}\}$$

$\gamma_j = \pm 1$ are the chiralities of the momenta$\{k\}, \{q\}$, $\gamma_i = 1$ for $\{k\}$ and $\gamma_j = -1$ for $\{q\}$. The $A_P(Q)$ are constants which satisfy

$$A_P(\ldots, Q_i, \ldots, Q_j, \ldots) = -P_{ij} A_P(\ldots, Q_j, \ldots, Q_i, \ldots) \tag{21}$$

for the Fermi statistics.

For an eigenstate, the functions $\Psi_{m_\uparrow, m_\downarrow}$ are determined by

$$A_{\ldots, P_i, \ldots, P_j, \ldots}(\ldots, Q_i, \ldots, Q_j, \ldots) = S_{r_i, r_j}(s_{P_i} - s_{P_j}) A_{\ldots, P_j, \ldots, P_i, \ldots}(\ldots, Q_j, \ldots, Q_i, \ldots) \tag{22}$$

$$\Psi_{m_\uparrow, m_\downarrow} = \sum_{Q,P} \prod_{l=1}^{m_\uparrow} R_{ij} S^{+}_{\uparrow,\uparrow}(k^{(l)} - q^{(l)}) \prod_{m=1}^{m_\downarrow} R_{i'j'} S^{+}_{\downarrow,\downarrow}(k^{(m)} - q^{(m)}) \\ \times \Psi_{0,0}[Q, P](\ldots, \{x_i = y_j = z_l\}, \ldots, \{x_{i'} = y_{j'} = w_m\}, \ldots) \tag{23}$$

where $k^{(l)}, q^{(l)}$ and $k^{(m)}, q^{(m)}$ are the momenta carried by the fermions at x_i, y_j and $x_{i'}, y_{j'}$ respectively; $\Psi_{0,0}[Q, P]$ is the value of $\Psi_{0,0}$ in the region $[Q, P]$. For details of the construction of (23) we refer the readers to [11]. The eigenvalue associated with the state is

$$E = v_{\mathrm{F}} \sum_{i=1}^{N_+} k_i - v_{\mathrm{F}} \sum_{j=1}^{N_-} q_j \, . \tag{24}$$

Below we use the periodic conditions

$$\Psi_{0,0}(\ldots, x_i, \ldots, y_j, \ldots) = \Psi_{0,0}(\ldots, x_i + L, \ldots, y_j, \ldots) = \Psi_{0,0}(\ldots, x_i, \ldots, y_j + L, \ldots) \tag{25}$$

to derive the Bethe ansatz equations. Obviously, the equations

$$\Psi_{m_\uparrow, m_\downarrow}(\ldots, x_i, \ldots, y_j, \ldots) = \Psi_{m_\uparrow, m_\downarrow}(\ldots, x_i + L, \ldots, y_j, \ldots) \\ = \Psi_{m_\uparrow, m_\downarrow}(\ldots, x_i, \ldots, y_j + L, \ldots) \tag{26}$$

are satisfied for $\Psi_{m_\uparrow, m_\downarrow}$ are nothing but the contraction of $\Psi_{0,0}$. According to Yang [12], the spectrum of the Hamiltonian is determined by the following eigenvalue problem:

$$\begin{aligned} \mathrm{e}^{\mathrm{i}k_i L} \zeta_0 &= S_{+,+}(k_i - k_{i+1}) \cdots S_{+,+}(k_i - k_{N_+}) S_{+,+}(k_i - k_1) \cdots S_{+,+}(k_i - k_{i-1}) \\ &\quad \times S_{+,-}(k_i - q_1) \cdots S_{+,-}(k_i - q_{N_-}) \zeta_0 \\ \mathrm{e}^{\mathrm{i}q_j L} \zeta_0 &= S_{-,+}(q_j - k_{i+1}) \cdots S_{-,+}(q_j - k_{N_+}) S_{-,+}(q_j - k_1) \cdots S_{-,+}(q_j - k_i) \\ &\quad \times S_{-,-}(q_j - q_1 \cdots S_{-,-}(q_j - q_{j-1}) \\ &\quad \times S_{-,-}(q_j - q_{j+1}) \cdots S_{-,-}(q_j - q_{N_-}) \zeta_0 \, . \end{aligned} \tag{27}$$

The above equations readily give the Bethe ansatz equations

$$\begin{aligned} \mathrm{e}^{\mathrm{i}k_i L} &= \prod_{i'=1}^{N_+} \exp[\mathrm{i}\phi_+(k_i - k_{i'})] \prod_{\alpha=1}^{M_\downarrow} \frac{k_i - \Lambda_\alpha - \frac{1}{2}\mathrm{i}c}{k_i - \Lambda_\alpha + \frac{1}{2}\mathrm{i}c} \prod_{j=1}^{N_-} \frac{k_i - q_j + \mathrm{i}c}{k_i - q_j - \mathrm{i}c} \\ \mathrm{e}^{\mathrm{i}q_j L} &= \prod_{j'=1}^{N_-} \exp[\mathrm{i}\phi_-(q_j - k_{j'})] \prod_{\alpha=1}^{M_\downarrow} \frac{q_j - \Lambda_\alpha - \frac{1}{2}\mathrm{i}c}{q_j - \Lambda_\alpha + \frac{1}{2}\mathrm{i}c} \prod_{i=1}^{N_+} \frac{q_j - k_i + \mathrm{i}c}{q_j - k_i - \mathrm{i}c} \\ \prod_{i=1}^{N_+} \frac{k_i - \Lambda_\alpha - \frac{1}{2}\mathrm{i}c}{k_i - \Lambda_\alpha + \frac{1}{2}\mathrm{i}c} \prod_{j=1}^{N_-} \frac{q_j - \Lambda_\alpha - \frac{1}{2}\mathrm{i}c}{q_j - \Lambda_\alpha + \frac{1}{2}\mathrm{i}c} &= -\prod_{\beta=1}^{M_\downarrow} \frac{\Lambda_\beta - \Lambda_\alpha - \mathrm{i}c}{\Lambda_\beta - \Lambda_\alpha + \mathrm{i}c} \end{aligned} \tag{28}$$

where L is the length of the system and $M_\uparrow \geqslant M_\downarrow$ is supposed.

3. The ground state

Different choices of $\phi_r(k)$ certainly give different physical states and even change the operator content of the theory [13–15]. In this paper, we consider only $\phi_r(k) = 0$ case.

From equations (28) we can see that k_i and q_j may have conjugate pair solutions when $L \to \infty$

$$k_i = \lambda_i \pm \tfrac{1}{2}\mathrm{i}c \qquad q_j = \omega_j \pm \tfrac{1}{2}\mathrm{i}c \qquad \{\lambda_i; \omega_j\} = \{\Lambda_\alpha\}. \tag{29}$$

We consider $N_+ = N_- = M_\uparrow = M_\downarrow$ case. For simplicity, we put N_+ being even. This allow us to obtain an unique ground state. The Bethe ansatz equations with equation (29) thus reduce to

$$\begin{aligned}
\mathrm{e}^{2\mathrm{i}\lambda_i L} &= -\prod_{j=1}^{N/4} \frac{\lambda_i - \lambda_j - \mathrm{i}c}{\lambda_i - \lambda_j + \mathrm{i}c} \prod_{l=1}^{N/4} \left\{ \frac{\lambda_i - \omega_l + \mathrm{i}c}{\lambda_i - \omega_l - \mathrm{i}c} \right\} \left\{ \frac{\lambda_i - \omega_l + 2\mathrm{i}c}{\lambda_i - \omega_l - 2\mathrm{i}c} \right\} \\
\mathrm{e}^{2\mathrm{i}\omega_l L} &= -\prod_{m=1}^{N/4} \frac{\omega_l - \omega_m - \mathrm{i}c}{\omega_l - \omega_m + \mathrm{i}c} \prod_{j=1}^{N/4} \left\{ \frac{\omega_l - \lambda_j + \mathrm{i}c}{\omega_l - \lambda_j - \mathrm{i}c} \right\} \left\{ \frac{\omega_l - \lambda_j + 2\mathrm{i}c}{\omega_l - \lambda_j - 2\mathrm{i}c} \right\}.
\end{aligned} \tag{30}$$

As the spectrum of the present model is not bounded from below, a cutoff should be used. We put $|\lambda_i|, |\omega_l| \leqslant K$. Asymptotically, we shall take the limit $K \to \infty$. To obtain the lowest energy state, we should choose $\lambda_i < 0$ and $\omega_l > 0$. In the thermodynamic limit, the ground state is described by the following integral equations:

$$\begin{aligned}
\rho_{+,g}(\lambda) &= \frac{1}{\pi} - \int_{-K}^{0} a_2(\lambda - \lambda')\rho_{+,g}(\lambda')\,\mathrm{d}\lambda' + \int_0^K [a_2(\lambda - \omega) + a_4(\lambda - \omega)]\rho_{-,g}(\omega)\,\mathrm{d}\omega \\
\rho_{-,g}(\omega) &= \frac{1}{\pi} - \int_0^K a_2(\omega - \omega')\rho_{-,g}(\omega')\,\mathrm{d}\omega' + \int_{-K}^{0} [a_2(\omega - \lambda) + a_4(\omega - \lambda)]\rho_{+,g}(\lambda)\,\mathrm{d}\lambda
\end{aligned} \tag{31}$$

where $\rho_{+,g}(\lambda)$ and $\rho_{-,g}(\omega)$ are the density distributions of λ and ω; the kernels $a_n(\lambda)$ are given by

$$a_n(\lambda) = \frac{1}{\pi} \frac{\frac{1}{2}nc}{\lambda^2 + (\frac{1}{2}nc)^2}. \tag{32}$$

The ground-state energy takes the form

$$E_g/L = 2\int_{-K}^{0} \lambda\rho_{+,g}(\lambda)\,\mathrm{d}\lambda - 2\int_0^K \omega\rho_{-,g}(\omega)\,\mathrm{d}\omega. \tag{33}$$

Thus the Fermi sea consists of all negative λ states and all positive ω states are filled up to the cutoff. The dressed energy [16] in our case satisfies

$$\begin{aligned}
\epsilon_+(\lambda) &= 2v_\mathrm{F}\lambda - \int_{-K}^{0} a_2(\lambda - \lambda')\epsilon_+(\lambda')\,\mathrm{d}\lambda' + \int_0^K [a_2(\lambda - \omega) + a_4(\lambda - \omega)]\rho_{-,g}(\omega)\,\mathrm{d}\omega \\
\epsilon_-(\omega) &= -2v_\mathrm{F}\omega - \int_0^K a_2(\omega - \omega')\epsilon_-(\omega')\,\mathrm{d}\omega' + \int_{-K}^{0} [a_2(\omega - \lambda) + a_4(\omega - \lambda)]\rho_{+,g}(\lambda)\,\mathrm{d}\lambda.
\end{aligned} \tag{34}$$

Note that the dressed energy is also unbounded from below when $K \to \infty$. However, the derivatives $\epsilon_\pm{}'$ and the densities $\rho_{\pm,g}$ are convergent for $K \to \infty$. This gives meaningful collective excitations with sound velocity [16]

$$v = \frac{\epsilon'{}_{+}(0)}{2\pi\rho_{+,g}(0)}\,. \tag{35}$$

4. Thermodynamics

We shall construct the thermodynamics of the present model by following the method developed by Yang and Yang [17] and Takahashi [18]. From the Bethe ansatz equations we can see that the solutions are grouped as Cooper pairs:

$$k_\alpha^{\pm} = \Lambda'_\alpha \pm \tfrac{1}{2}\mathrm{i}c \qquad q_\beta^{\pm} = \Lambda_\beta\text{"} \pm \tfrac{1}{2}\mathrm{i}c \tag{36}$$

real k, q and Λ strings

$$\Lambda^n_{\gamma,j} = \Lambda^n_\gamma + \tfrac{1}{2}\mathrm{i}c(n+1-2j) \qquad j = 1, 2, \ldots, n \tag{37}$$

where Λ'_α, Λ''_β and Λ^n_γ are real. Substitute the above solutions into equations (28) and take the thermodynamic limit $L \to \infty, N \to \infty$. We obtain the following integral equations:

$$\begin{aligned}
&\rho_1(k) + \rho_1^h(k) = \frac{1}{2\pi} - [1]\rho_{1s}(k) + [3]\rho_{2s}(k) + [2]\rho_2(k) - \sum_{n=1}^{\infty}[n]\sigma_n(k)\\
&\rho_2(q) + \rho_2^h(q) = \frac{1}{2\pi} - [1]\rho_{2s}(q) + [3]\rho_{1s}(q) + [2]\rho_1(q) - \sum_{n=1}^{\infty}[n]\sigma_n(q)\\
&\rho_{1s}(k) + \rho_{1s}^h(k) = \frac{1}{\pi} - [2]\rho_{1s}(k) + A_{13}\rho_{2s}(k) + [3]\rho_2(k) - [1]\rho_1(k)\\
&\rho_{2s}(q) + \rho_{2s}^h(q) = \frac{1}{\pi} - [2]\rho_{2s}(q) + A_{13}\rho_{1s}(q) + [3]\rho_1(q) - [1]\rho_2(q)\\
&[n]\{\rho_1(\Lambda) + \rho_2(\Lambda)\} = \sigma_n^h(\Lambda) + \sum_{m=1}^{\infty} A_{nm}\sigma_m(\Lambda)
\end{aligned} \tag{38}$$

where ρ_i, ρ_i^h denote the densities of $k(q)$ and $k(q)$ holes, σ_n, σ_n^h denote the densities of the n string and n-string holes, ρ_{is}, ρ_{is}^h denote the density distributions of the Cooper pairs, $[n]$ is a integral operator with the kernel $a_n(k-k')$ and

$$A_{nm} = [|m-n|] + 2[|m-n|+2] + \cdots + 2[|m+n-2|] + [m+n]\,. \tag{39}$$

Note that k, q take values in the interval $[-K, K]$ and Λ takes values in the interval $(-\infty, \infty)$. After some manipulations we get the thermo-potential as

$$\Omega/L = -\frac{T}{2\pi}\sum_{i=1}^{2}\int \ln\{1+\zeta_i^{-1}(k)\}\,\mathrm{d}k - \frac{T}{\pi}\sum_{i=1}^{2}\int \ln\{1+\zeta_{is}^{-1}(k)\}\,\mathrm{d}k \tag{40}$$

where the $\zeta_i(k), \zeta_{is}(k)$ are determined by the following integral equations:

$$
\begin{aligned}
\ln\zeta_1(k) &= \frac{v_{\rm F}k - H - A}{T} - [2]\ln\{1+\zeta_2^{-1}(k)\} + [1]\ln\{1+\zeta_{1s}^{-1}(k)\} \\
&\quad -[3]\ln\{1+\zeta_{2s}^{-1}(k)\} - \sum_n [n]\ln\{1+\eta_n^{-1}(k)\} \\
\ln\zeta_2(k) &= \frac{-v_{\rm F}k - H - A}{T} - [2]\ln\{1+\zeta_1^{-1}(k)\} + [1]\ln\{1+\zeta_{2s}^{-1}(k)\} \\
&\quad -[3]\ln\{1+\zeta_{1s}^{-1}(k)\} - \sum_n [n]\ln\{1+\eta_n^{-1}(k)\} \\
\ln\zeta_{1s}(k) &= \frac{2(v_{\rm F}k - A)}{T} + [1]\ln\{1+\zeta_1^{-1}(k)\} + [2]\ln\{1+\zeta_{1s}^{-1}(k)\} \\
&\quad -[3]\ln\{1+\zeta_{2s}^{-1}(k)\} - A_{13}\ln\{1+\zeta_{2s}^{-1}(k)\} \\
\ln\zeta_{2s}(k) &= \frac{-2(v_{\rm F}k + A)}{T} + [1]\ln\{1+\zeta_2^{-1}(k)\} + [2]\ln\{1+\zeta_{2s}^{-1}(k)\} \\
&\quad -[3]\ln\{1+\zeta_{1s}^{-1}(k)\} - A_{13}\ln\{1+\zeta_{1s}^{-1}(k)\} \\
\ln\{1+\eta_n(\Lambda)\} &= \frac{2nH}{T} + \sum_{m=1}^{\infty} A_{nm}\ln\{1+\eta_m^{-1}(\Lambda)\} \\
&\quad +[n]\{\ln\{1+\zeta_1^{-1}(\Lambda)\} + \ln\{1+\zeta_2^{-1}(\Lambda)\}\}
\end{aligned}
\tag{41}
$$

with the boundary condition

$$
\lim_{n\to\infty}\{[n+1]\ln(1+\eta_n) - [n]\ln(1+\eta_{n+1})\} = -\frac{2H}{T} \tag{42}
$$

where H and A are the magnetic field and the chemical potential, respectively. Equations (41) with the condition (42) is closed. From equation (41) we can see that $\zeta_1(k) = \zeta_2(-k), \zeta_{1s}(k) = \zeta_{2s}(-k)$. This may simplify equations (41).

For the $H = 0$ and $T = 0$ case, the spin degrees of freedom seem to be frozen via the spin gap of the pair states. The charge sector of the system at zero temperature is conformally invariant with the conformal anomaly $c = 1$. According to the predictions of the conformal field theory [19], the density of the free energy at low temperatures takes the following form:

$$
f(T) = \frac{\pi T^2}{6v} + {\rm o}(T^2) \tag{43}
$$

where v is the sound (plasmon) velocity given by equation (35).

5. Conclusion

In this paper we consider a one-dimensional t-channel boson–fermion model. It is shown that this model can be solved via Bethe ansatz. The electrons in different chiral branches may form Cooper-pair states with proper choices of the phase factor ϕ_r. This is not very strange because in the Bethe ansatz solvable models, the bound states usually correspond to the conjugate complex roots of the Bethe ansatz equations. This means the Cooper pair must be formed by two electrons near the same Fermi point. The situation of our model is very similar to that of the multi-channel Kondo problem in which the flavour bound states are formed via dynamics [20].

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